Lecture 3

Measures

A set X w/ σ-algebra $\mathcal{M}$ is called a <u>measurable space</u>.

A <u>measure</u> on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ s.t.

(i) $\mu(\emptyset) = 0$

(ii) $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$ when

$$E_k \cap E_\ell = \emptyset \text{ for } k \neq \ell.$$

If one instead of (ii) requires

(ii) $\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$, $E_k \cap E_\ell = \emptyset$.

then one obtains a <u>finitely additive measure</u>.

More terminology.

- μ is finite if $\mu(X)<\infty$
- μ is σ-finite if $X=\bigcup_{n=1}^{\infty} E_n$

  and $\mu(E_n)<\infty$.
- μ is semifinite if $\forall E\in\mathcal{M}$ $\exists F\in\mathcal{M}$ s.t. $F\subseteq E$ and $0<\mu(F)<\infty$.

Construction of useful measures (like Lebesgue) require some work. At this point, we can give 2 trivial examples on $(X,\mathcal{P}(X))$:

① Counting measure. Let $\mu(E)=$ # of points in E ($=\infty$ if E not finite).

(2) Point mass (Dirac Delta) at $x_0 \in X$.

$$\mu(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E. \end{cases}$$

Basic facts.

Thm. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then (all sets below assumed in $\mathcal{M}$):

(i) (Monotonicity) $E \subset F \Rightarrow \mu(E) \leq \mu(F)$

(ii) (Subadditivity) $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$

(iii) (Cont. from below)

$$E_1 \subseteq E_2 \subseteq \dots \Rightarrow \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$$

(iv) (Cont. from above)

$$E_1 \supseteq E_2 \supseteq \dots \text{ and } \mu(E_1) < \infty$$

$$\Longrightarrow$$

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$$

Pf. (i), (ii) are DIY.

(iii). Let $F_1 = E_1$, $F_k = E_k \setminus E_{k-1}$, $k \geq 2$.

Then $F_k \in \mathcal{M}$ and $F_k \cap F_l = \emptyset$, $k \neq l$.

Moreover, $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} F_k$. Thus,

$$\mu(E) = \sum_{k=1}^{\infty} \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k)$$

But $\bigcup_{k=1}^{n} F_k = E_n \Rightarrow \mu(E) = \lim_{n\to\infty} \mu(E_n)$.

(iv) Let $F_1 = \emptyset$, $F_k = E_1 \setminus E_k$, $k \geq 2$.

Then, $E = \bigcap_{k=1}^{\infty} E_k = E_1 \setminus \left(\bigcup_{k=1}^{\infty} F_k \right)$.

$\Rightarrow E_1 = E \cup \left(\bigcup_{k=1}^{\infty} F_k \right)$ disjoint union. Moreover, $F_1 \subseteq F_2 \subseteq \dots$, so by (iii) $\Rightarrow$

$$\mu(E_1) = \mu(E) + \lim_{n\to\infty} \mu(F_n)$$

Since $\mu(E_1) = \mu(F_n) + \mu(E_n)$ and $\mu(E_1) < \infty \Rightarrow \mu(F_n) = \mu(E_1) - \mu(E_n)$

$$\Rightarrow \mu(E) = \lim_{n\to\infty} \mu(E_n). \quad \blacksquare$$

<u>Completions</u>.

If $(X, \mathcal{M}, \mu)$ is a measure space, and $E \in \mathcal{M}$ + $\mu(E) = 0$, then E is a <u>null set</u>. Thus, if $F \subseteq E$ <u>and $F \in \mathcal{M}$</u>, then $\mu(F) = 0$ so F is null. However, in general not all subsets of null sets need to be in $\mathcal{M}$.

<u>Def. 1</u>. A measure space $(X, \mathcal{M}, \mu)$ is complete if all subsets of null sets are measurable.